Algeblocks® Promote Algebraic Understanding

by Ferdinand D. Rivera

Department of Mathematics
San José State University

In the fall of 2007, I decided to teach Algebra 1 differently. At the very least, I wanted my students to *experience* firsthand with manipulatives what it meant to perform basic operations on simple algebraic expressions and solve linear and quadratic equations using concrete materials. The learning goal was very clear and nothing less: I wanted students to draw on their hands-on experiences with manipulatives in thinking and logically transitioning to a more formal understanding of the target concepts and processes. My decision to approach algebra learning using manipulatives, I should note, was not a novel one. Early algebra history did not start with the use of variables that were isolated from any specific, daily context, which is how students today typically learn Algebra 1 in the classroom. Early algebra understanding initially emerged from problems that were geometric and practical in contexts. For example, Walter Eells (1977) wrote about how the ancient Greeks learned to approximate the positive values of certain square roots and solve simple quadratic equations geometrically with the aid of line segments and rectangles. Bartel Van der Waerden (1985) illustrated how Al-Khwarizmi, an early Arab mathematician regarded today as the Father of Algebra, produced an algebraic method for solving quadratic problems that were initially about dividing squares proportionally.

In my Algebra 1 class of 34 eager Grades 7 and 8 students in an urban school in Northern California, we began the school year by familiarizing ourselves with ETA/Cuisenaire®'s Algeblocks®, a three-dimensional set of geometric prisms that model two-variable polynomial expressions of up to the third power (www.Algeblocks.com). In **Figure 1**, students saw useful visual models for the following basic algebraic terms: xy, x^2, y^2, yx^2, xy^2, x^3, and y^3.

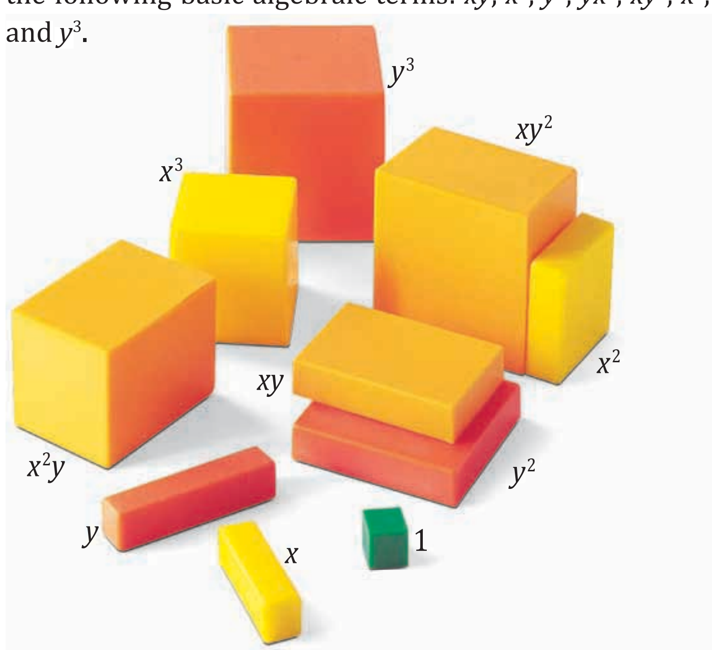


Figure 1: *Algeblocks for basic polynomial terms*

Three things are worth noting:

First, the introduction to Algeblocks helped my students review their concepts of areas and volumes, including properties of prisms.

Second, the prisms enabled students to infer why x^3 and $3x$ or y^2 and $2y$ were different expressions for almost all values of x and y, respectively. When they gathered an x^3 block and three of the x blocks, corresponding to the expression $3x$, the kinesthetic and visual experience allowed them to compare and verify firsthand why the expressions were different and not *like* terms. We then repeated the same Algeblocks activity of comparing several other different pairs of Algeblocks in order to help students overcome what previous research studies have noted about students' misconceptions about, and difficulties in understanding, algebraic expressions.

Third, we are now entering the beginning phase of the Common Core standards (www.corestandards.org/the-standards/mathematics). In Algebra 1, the document is clear about providing students with every opportunity to develop the *mathematical practice* of *seeing structures in expressions*, which involves *interpreting the structure of expressions* and *writing expressions in equivalent forms to solve problems*. Approaching Algebra 1 using Algeblocks certainly supports this particular Common Core recommendation.

In the remainder of this article, I share my learning experiences as I used Algeblocks in assisting my Algebra 1 students to develop valid *logicomathematical knowledge*. Such knowledge is synonymous with constructed conceptual relationships that are inferred on objects as a consequence of seeing and establishing similarities and differences. In educational terms, logicomathematical knowledge describes the abstract phase in the Concrete–Representational–Abstract (C-R-A) continuum of learning. Let's try the following activities below:

Activity 1. Place an orange cube (y^3) and a yellow cube (x^3) next to each other. Then, gather a yellow rectangular prism (x) and an orange rectangular prism (y) and keep them together. Why are they labeled that way and not some other way? Do you notice anything similar and/or different within each pile? How about across the piles?

Activity 2. Gather several blocks of x^3. How might we make sense of $5x^3 + 4x^3$? Experiment with the Algeblocks. How might you express the result?

Activity 3. Gather the blocks corresponding to the expression $2x^3 + 4y^3 - x^3 + y^3$ on the Basic Mat. How might we simplify it? What does it mean, then, to simplify a series of polynomial expressions involving addition and subtraction?

Activity 4. There are no Algeblocks for powers greater than 3. Why do you think so? We can, of course, logically infer a structural relationship for higher orders on the basis of our visual experiences with the Algeblocks. Having done Activities 2 and 3, how might we extend the visually drawn process in dealing with polynomial expressions of higher powers, say, x^4, $2x^4 + 5x^4$, $3x^4 - 2x^3 + 7x^4 - 4x^3$?

In class, if students could only remember seeing different colors or shapes, then all they are learning is *physical knowledge* of the Algeblocks, and this is not the state that we want them to achieve at the end of a session with Algeblocks, or with any manipulatives. The concrete phase in C-R-A would have students thinking in terms of mathematical relationships beyond physical knowledge. For example, when they could see beyond the colors and shapes and are able to infer that, say, x^3 and y^3 are two different expressions, assuming different values for x and y, then students have concrete knowledge. Further, when students are able to make sense of mathematical ideas such as *like* and *unlike* terms in algebra, then they have abstract knowledge in addition to concrete knowledge.

The preceding four activities capture the essence of the title of this essay. The wonderful thing about using Algeblocks in the classroom is that these manipulatives enable students to experience what it means to construct meaningful knowledge following the C-R-A model. Certainly, what is ultimately desired is abstract knowledge; that is, concrete hands-on experiences with the Algeblocks should allow students to establish relevant and valid mathematical relationships and structures. Drawing on my own experiences with my Algebra 1 class, learning with the Algeblocks allowed them easy access to algebraic ideas that are in some cases hard to reach and counterintuitive.

Algeblocks-Driven Insights into Basic Operations with Polynomials

In my Algebra 1 class, students studied polynomials in the beginning of the school year. They initially performed adding, subtracting, multiplying, and dividing polynomial expressions with Algeblocks. What was relevant about the Algeblocks-driven adding and subtracting polynomials was the natural context in which students learned about the meaning and significance of the concept of a *unit*, which is central to one's understanding of *like* and *unlike* terms in algebra. What does it mean? Let's try Activity 5.

Activity 5. Gather the Algeblocks you need to complete the activity shown on **Figure 2**. Then answer these questions: When are two or more algebraic expressions said to be *like* terms? When are they said to be *unlike*? What does it mean to simplify polynomial expressions? Whenever we add or subtract *like* terms, which factors stay the same, and which change?

The quadrant mat and the factor track are extremely useful components of Algeblocks. They provide a concrete context that enables students to understand processes of multiplication and division involving simple polynomials.

Activity 6. Set up the quadrant mat and the factor track to help you make sense of the famous formula for the square of a binomial, $(x + y)^2 = x^2 + 2xy + y^2$. The resulting expression $x^2 + 2xy + y^2$ is called a perfect square trinomial. Algeblocks enables students to see why the expression is called a perfect square trinomial, thus, allowing students to build an understanding of correct mathematical vocabulary. Now ask yourself why $(x + y)^2 \neq x^2 + y^2$ for nonzero values of x and y with the Algeblocks.

Activity 7. Set up the quadrant mat and the factor track to obtain the quotient when −10 is divided by −2. Do the same in the case of $(3xy - 6x)$ upon division by $3x$. Now ask yourself what division might mean in the context of this problem-solving experience.

In my class, division of polynomials, which includes integers, took place within the Algeblocks context of finding the missing dimension when the area of a rectangle and one of its dimensions were both known. Students became very interested in the process of factoring because the concrete experiential activity enabled them to visually represent the process from both geometric and algebraic perspectives. Relative to Activity 6, because students spent time exploring the factored form of perfect square trinomials, they used their visual experiences with the Algeblocks in establishing a generalization of the corresponding factored form, $(x + y)^2$. Later students learned to factor general quadratic trinomials using this same type of visual strategy.

Activity 8. Set up the quadrant mat and the factor track. Lay out the blocks corresponding to $x^2 + 6x + 9$ and try to reconfigure them to form a rectangle. If it is possible, what are the dimensions of this rectangle? Do the same for the following expressions: $2x^2 + 5x + 2$; $x^2 - 6x + 6$.

In the above activity, when my students saw that the Algeblocks corresponding to each expression could be conveniently reconfigured into a rectangle on the mat so that the factors could be easily read from the track, they experienced an AHA! moment. In an assessment context,

Example: Simplify: $x^2 - 4x + 2x^2 + 2x$

Step 1.	Step 2.	Step 3.	Step 4.
Model the entire expression on the mat.	Combine like terms. Remove zero pairs.	Read the mat. 3 of x^2 and $^-2$ of x	Record. $3x^2 - 2x$

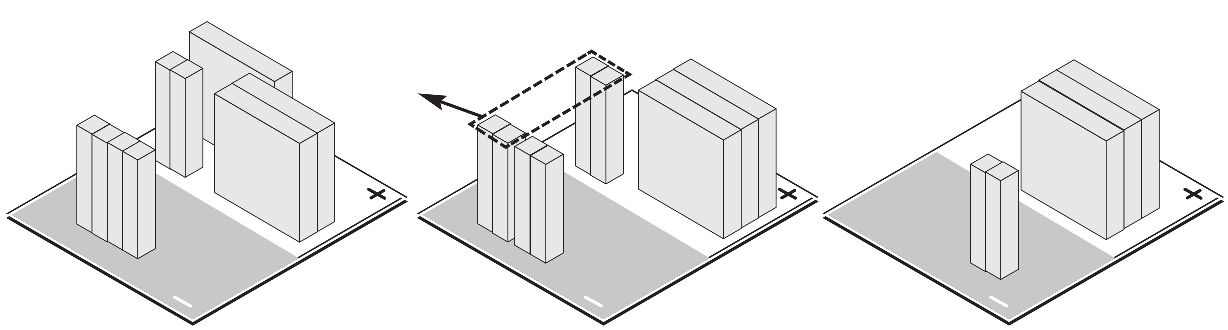
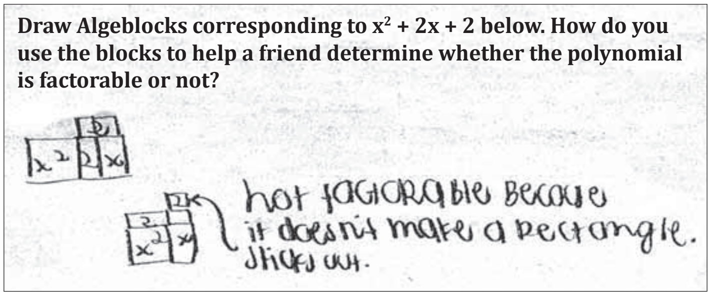


Figure 2: *Activity 5*

students Cheska and Jamal in **Figure 3** understood why and how it was not possible to factor some quadratic trinomial expressions (over, say, the set of integers). In terms of the C-R-A model of learning, Cheska and Jamal produced their own pictorial representations that have been drawn from their concrete experiences with Algeblocks.

Cheska



Jamal

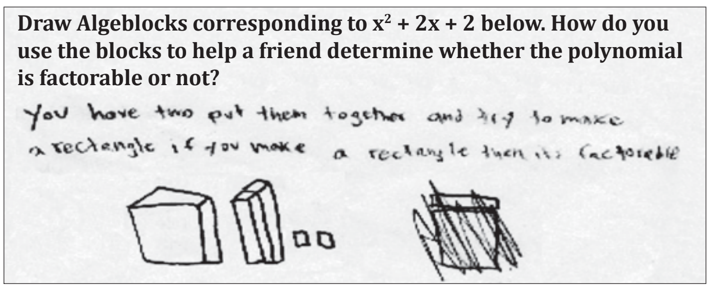


***Figure 3:** Algebra 1 students' written responses concerning the nonfactorability of $x^2 + 2x + 2$*

Solving Quadratic Equations

One important algorithm that my Algebra 1 class learned was the quadratic formula. We used the formula in cases where the roots of the quadratic equation $ax^2 + bx + c$, where $a \neq 0$, involved unfriendly rational numbers, irrational numbers, and imaginary numbers. Most everything else was easy for them. Repeated experiences with the factor track and the quadrant mat enabled students to develop their factoring skills, which then helped them in obtaining the necessary roots of the equation or zeros in the context of quadratic functions. How might we use Algeblocks to help students learn the famous formula? Start with Activity 9 below.

Activity 9. Gather two piles of blocks with one pile showing the sum ($x^2 + 4x$) and the other pile containing 8 green unit cubes. Assuming the two piles have equal value, this means we need to solve the quadratic equation $x^2 + 4x = 8$. Like any problem with equations, what does it mean to solve for x? That is, how do we isolate x on one side of the equation especially in this case?

Activity 9 describes how I opened up the discussion in my Algebra 1 class. At the outset, we do not want our students to model the incorrect solution shown in **Figure 4** *in which Emma tried to solve for n by over-generalizing the process for solving an unknown in a linear equation ax + b = c.*

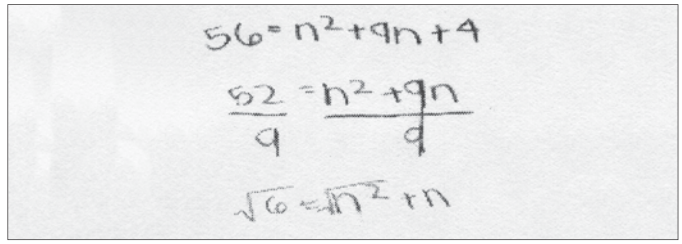


***Figure 4:** Emma's work on solving a quadratic equation*

Going back to Activity 9, organize the Algeblocks on the left pile to look like a rectangle (**Figure 5**). Notice that the dimensions of this rectangle are x and $(x + 4)$.

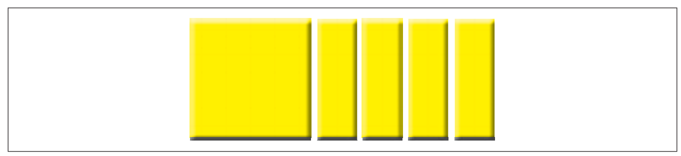


***Figure 5:** Algeblocks representation for $x^2 + 4x$*

Now, reconfigure the rectangle so that it looks like a square. How do we do this? First, take half the x blocks and gather them under the other side of the square. You should have something like the one in **Figure 6***.*

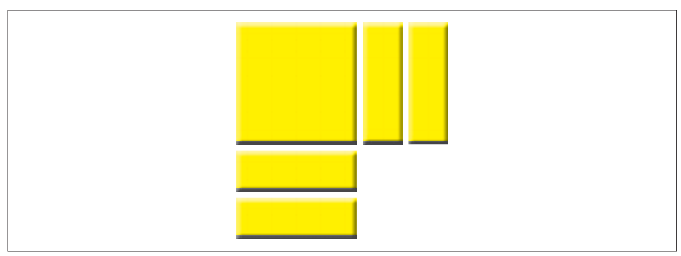


***Figure 6:** Reconfigured incomplete square*

Students should see that taking half the x blocks actually helps in forming a square. In **Figure 5***, the emerging dimensions of the reconfigured square appear to be (x + 2) by (x + 2). What remains to be done is to* **complete the square** *by adding 2^2 or 4 green cubes (***Figure 7***).*

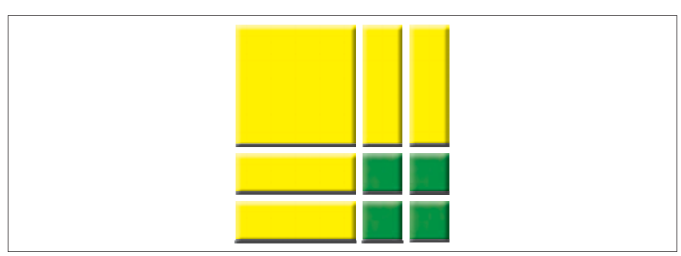
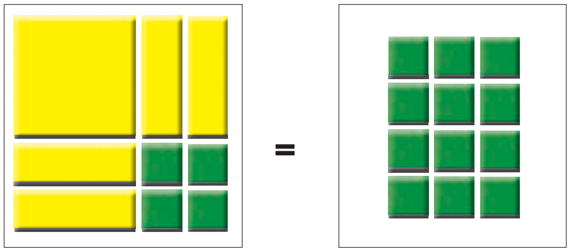


***Figure 7:** Addition of 4 green cubes to complete the square in Figure 4*

Now, since we added 4 positive green cubes on the (x^2 + 4x)-pile, we ought to add the same number of positive green cubes on the existing pile of 8 green cubes for a total of 12 positive green cubes (**Figure 8**).



$x^2 + 4x + 4 \quad = \quad 12$

Figure 8: *The completed square on the left and the 12 cubes on the right*

To solve for x, notice that

$$(x+2)^2 = \left(\pm\sqrt{12}\right)^2.$$

Keep in mind that we are solving for x regardless of context, which means we are not restricted to positive values alone. We can, thus, conclude that

$$x + 2 = \pm\sqrt{12}.$$

So, $x = -2 \pm \sqrt{12}$.

In my class, we employed the above concrete process a several more times with Algeblocks and paid close attention to the following three crucial steps below.

Algeblocks Actions

A. Take half the pile of linear blocks and reconfigure the rectangle to form an incomplete square (e.g., Figure 6).

B. Complete the bigger square by adding a smaller square of green cubes whose dimensions depend on what is known in action A (e.g., Figure 7).

C. Add the same amount of green cubes on the other pile of green cubes and then compare the dimensions of the two squares (e.g., Figure 8).

Algebraic Actions on $x^2 + bx = c$

A′. Take half the coefficient of the linear term, $\frac{b}{2}$.

B′. $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

C′.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = \frac{4c + b^2}{4}$$

$$x + \frac{b}{2} = \pm\sqrt{\frac{4c + b^2}{4}}$$

Because my students visually understood the above concrete steps, transitioning to the more general situation involving the equation $ax^2 + bx + c = 0$ became an easy and meaningful experience for them, as shown below. Drawing on their experiences with the Algeblocks, they understood why it was convenient to initially divide the equation by the nonzero coefficient a in step III. They also saw the central role of completing the square in *isolating the variable x in steps IV and V. Further, they saw the usefulness of the identity* $\sqrt{a^2} = |a| = \begin{cases} a, & a \ge 0 \\ -a, & a < 0 \end{cases}$ *that enabled them to infer the existence of two possible values for the expression* $x + \frac{b}{2a}$ *in step VI.*

$$\text{I.} \quad ax^2 + bx + c = 0$$

$$\text{II.} \quad ax^2 + bx = -c$$

$$\text{III.} \quad x^2 + \frac{b}{a}x = \frac{-c}{a}$$

$$\text{IV.} \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\text{V.} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$\text{VI.} \quad x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}}$$

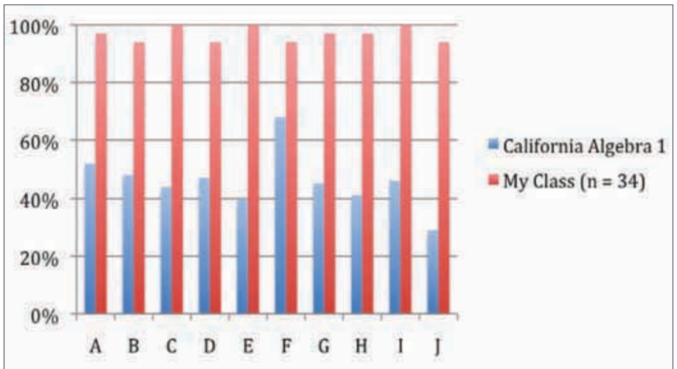
$$\text{VII.} \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{VIII.} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As a consequence of the above Algeblocks activity, my students did not have to force themselves to memorize the quadratic formula in step VIII since the concrete experience enabled them to construct an image of the necessary steps in completing the square leading to the solution of any quadratic equation. Also, they were able to connect the vocabulary to the process that further strengthened the use of correct mathematical language.

Performance Data Results Drawn from the Algebra 1 Class

So, how did my Algebra 1 class do in both local and state assessments? **Diagram 1** shows how my students performed on a 10-item classroom unit test that assessed basic aspects of the quadratic, polynomial, and rational expressions strand of the California Algebra 1 standards. The items were drawn from the state's released item bank, which also provides mean percentages of correct responses across all Algebra 1 students tested in California in previous years. Overall, my students' mean performance (97%) on those tasks was significantly better than the state mean (46%).



Questions
(Appeared in a multiple-choice format but the choices were deleted due to space limits)

A. What is a factor of $x^2 - 11x + 24$?

B. What are the solutions for the quadratic equation $x^2 + 6x = 16$?

C. $(4x^2 - 2x + 8) - (x^2 + 3x - 2) = ?$

D. Which is the factored form of $3a^2 - 24ab + 48b^2$?

E. Which of the following expressions is equal to $(x + 2) + (x - 2)(2x + 1)$?

F. The sum of two binomials is $5x^2 - 6x$. If one of the binomials is $3x^2 - 2x$, what is the other binomial?

G. What is $\frac{x^2 - 4xy + 4y^2}{3xy - 6y^2}$ reduced to lowest terms?

H. Simplify $\frac{6x^2 + 21x + 9}{4x^2 - 1}$ to lowest terms.

I. What is $\frac{x^2 - 4x + 4}{x^2 - 3x + 2}$ reduced to lowest terms?

J. $\frac{7z^2 + 7z}{4z + 8} \cdot \frac{z^2 - 4}{z^3 + 2z^2 + z} =$

Diagram 1: *Classroom Assessment Data Drawn from My Algebra 1 Class*

The California Algebra 1 state examination consists of 65 items and targets the following clusters of topics (following test blueprint): number properties, operations, and linear equations (26%); graphing and systems of linear equations (22%); quadratics and polynomials (32%); and functions and rational expressions (20%). In my class, we used Algeblocks to make sense of concepts and processes relevant to linear and quadratic equations and quadratic, polynomial, and rational expressions. **Diagram 2** shows the prior-year state results of my Algebra 1 class and the rest of the students in the same school who officially registered to take Algebra 1. At the beginning of the year, only about 65% of my students met grade-level standards (i.e., those who scored proficient and advance) compared with about 89% from the other group.

Diagram 3 shows the state results in Algebra 1 at the end of the year for both groups. Overall, my class performed significantly better than the other group, which I attribute, without a doubt, to my students' consistent and positive experience with Algeblocks.

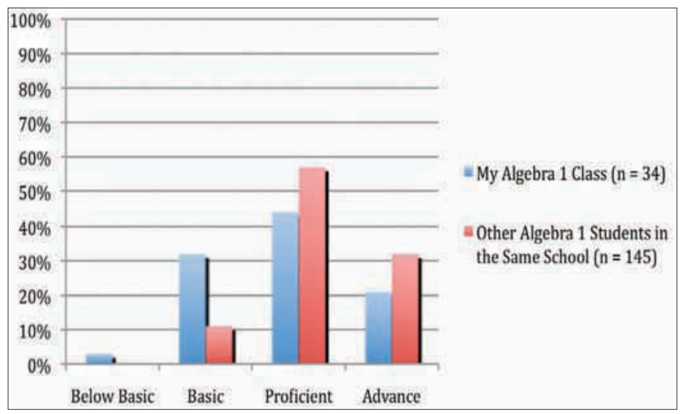


Diagram 2: *Prior Year STAR Results of My Class and Other Algebra 1 Classes in the Same School Before the Algeblocks-Driven Instruction*

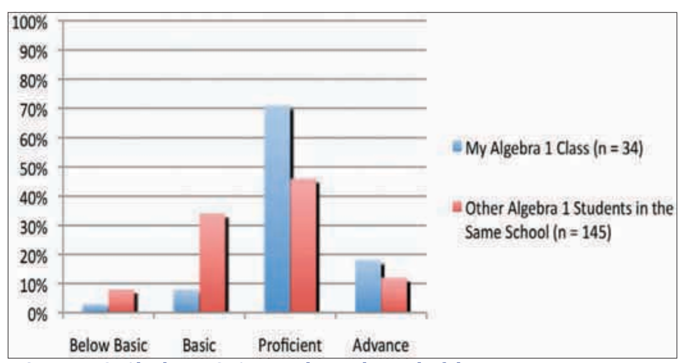


Diagram 3: *Algebra 1 STAR Results at the End of the Year*

Meaningful Construction of Logicomathematical Knowledge: The Key Is Manipulatives

Algeblocks and, more generally, manipulatives, are tools to think with, and to help students make sense of the mathematics they often encounter through abstract presentations. Students will most likely have fun using manipulatives to learn concepts and processes, but they also need to use them to acquire abstract knowledge. If used effectively, Algeblocks can support student thinking in various phases of learning, from the concrete to the representational, and finally, to the abstract phase. Physical knowledge drawn from concrete experiences such as Algeblocks and logicomathematical knowledge should go together. Further, while having concrete knowledge helps support memory recall and later, in producing pictorial representations, having abstract knowledge is a signpost that that the relevant logicomathematical relationships and properties have been acquired and understood.

In closing this reflective article, I offer the following advice for Algebra 1 teachers who are contemplating using Algeblocks in their own classrooms:

Use Algeblocks as manipulatives to think with and always with C-R-A in mind. Design individual and group interactions with students so that they are able to *think through doing*. Keep the presentation simple enough; do not distract them with large numbers of blocks that would be too difficult to manage. The most important thing to keep in mind is to see to it that students manifest purposeful, abstracting actions since none of the targeted mathematical relationships or structures inherently resides in Algeblocks. Students need to infer relationships and structures by performing actions on Algeblocks.

Use Algeblocks as a mediating tool with an eye on developing mathematical meaning and gaining entry into hard-to-reach concepts and processes. Bear in mind that Algebra 1 students are learning formal algebra for the first time, and what may seem easy to teachers does not necessarily mean easy to them. In fact, as shown in Diagram 1, on the previous page, only about 46% of Algebra 1 students in California could calculate the typical problems correctly. In my Algeblocks-mediated class, the average success rate of 97% could be attributed to the important role played by Algeblocks in assisting students with potential difficulties with both the alphanumeric symbols and the relevant concepts and processes. With this in mind, teachers need to allow their students to feel and explore interesting relationships and structures coupled with a consistent reminder about the necessity of establishing mathematical validity as well.

Minimize the use of Algeblocks when students have successfully transitioned to the abstract phase. In my class, those students who achieved full understanding of the relevant concepts and processes oftentimes manifested *visual fading* in their written work. That is, their visualizing slowly decreased over time and was replaced by alphanumeric processing. This learning condition meant that they had successfully experienced C-R-A. Teachers should also be keen to observe such changes in their students' thinking.

Finally, if learning does not proceed as planned, do not give up and resort to placing the blame on Algeblocks. Setting up the algebra classroom for Algeblocks intervention takes time and sustained effort, especially in the first year. However, based on my own experiences, it is all worth it in the long haul. In my Algebra 1 class, the students used them in—

- Explaining and illustrating mathematical ideas.
- Transitioning from the concrete to the representational and finally to the abstract phase of mathematical learning, the domain of logicomathematical knowledge.
- Gaining insights that could not be done with alphanumeric symbols alone.
- Reifying—*making objects come alive*—that made algebra learning fun, interesting, and worthwhile.

Algeblocks are useful tools that have purpose and value but only when students engage with them appropriately. While there is no royal road to learning algebra, the visual route through Algeblocks can definitely help ease and smooth the process.

References

Kamii, C., B. Lewis, and L. Kirkland. 2001. Manipulatives: When are they useful? Journal of Mathematical Behavior 20: 21–31.

Katz, V. 1997. Algebra and its teaching: A historical survey. Journal of Mathematical Behavior 16: 25–38.

Eells, W. 1977. Greek methods of solving quadratic equations. In S. Montgomery, E. Ralston, S. Gordon, G. Janusz, M. Schacher, & M. Smith (eds.), Selected papers on algebra: Volume 3 (pp. 444–463). Washington, DC: Mathematical Association of America.

Rivera, F. 2010. Toward a visually-oriented school mathematics curriculum: Research, theory, practice, and issues (Mathematics Education Library series 49). New York: Springer.

Rivera, F. 2010. There is more to mathematics than symbols. Mathematics Teaching 218: 42–47.

Uttal, D., K. Scudder, and J. DeLaoche. 1997. Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. Journal of Applied Developmental Psychology 18: 37–54.

Van der Waerden, B. 1985. A history of algebra: From Al-Khwarizmi to Emmy Noether. New York: Springer-Verlag.

About the Author

Ferdinand D. Rivera is an Associate Professor in the Department of Mathematics at San José State University, CA. He earned a Ph.D. in mathematics education and cultural studies at Ohio State University, and did post-doctoral work in Bonn, Germany. He taught high school mathematics for 10 years. Through a prestigious five-year National Science Foundation (NSF) Career Grant, he engaged in several year-long collaborative teaching projects with elementary and middle school teachers in their classrooms. Highlights of this paper have been drawn from his NSF-funded work teaching an Algebra 1 class for one year.

Dr. Rivera conducts empirical studies in algebraic thinking and, more generally, in mathematical cognition. His work has been published in several academic and professional journals for mathematics education researchers and practitioners. For a sample of his classroom-based research findings, readers are encouraged to reference his first book, *Toward a Visually Oriented School Mathematics Curriculum: Research, Theory, Practice, and Issues.*

Dr. Rivera also has conducted state-funded professional development programs and courses for middle and high school teachers in several California school districts. He may be reached at Ferdinand.Rivera@sjsu.edu.